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Positive Cyclic Systems of Linear Differential Equations

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1. INTRODUCTION

By a *positive cyclic system* of linear differential equations, we mean [1] a system of the form

$$dx_i/dt = p_i(t)\, x_{i+1}\,, \qquad p_i(t) > 0, \qquad i = 1, 2,..., n \tag{1}$$

(all subscripts taken modulo n here and throughout the paper). We shall assume throughout that the $p_i(t)$ are piecewise continuous. The system (1) will be called *uniformly positive* (u. p.) when

$$0 < m \leqslant p_i(t) \leqslant M = Nm < +\infty \quad \text{for} \quad i = 1, 2,..., n. \tag{1'}$$

A study of third-order, uniformly positive cyclic systems of linear differential equations was made in [1], with special reference to oscillation and order-of-growth theorems. In the present paper, we show that many of the conclusions hold, in modified form, for arbitrary n—but that the cases of even n and odd n are essentially different, and the case $n = 4$ has quite special properties.

For any n, the coordinate planes $x_i = 0$ separate $\mathbf{x}$-space into 2^n open *orthants*, each of which has a well-defined *signature* (sgn x_1 ,..., sgn x_n). Thus, the positive orthant $\mathscr{P}$ has the signature $(+,..., +)$, the negative orthant $-\mathscr{P}$ has the signature $(-,..., -)$, and when n is even, the *alternating orthant* $\mathscr{Q}$ has the signature $(+, -,..., +, -)$.

DEFINITION. A solution $\mathbf{x}(t)$ of (1) is said to be *oscillatory* (as $t \uparrow$) when its signature changes infinitely often on one, and, hence, on every interval $[a, +\infty)$; it is said to be oscillatory as $t \downarrow$ when its signature changes infinitely

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often on some interval $(-\infty, a]$. A solution of (1) which is not oscillatory is called ***nonoscillatory*** (as $t \uparrow$ or as $t \downarrow$, respectively). It is known [1] that any u. p. system (1), i.e., any system which satisfies (1)–(1′), has a projectively unique positive solution, which is clearly nonoscillatory. A closed orthant $\mathscr{O}$ is called *stable* as t increases [decreases] when, for any solution $\mathbf{x}(t)$ of (1), $\mathbf{x}(a) \in \mathscr{O}$ implies $\mathbf{x}(t) \in \mathscr{O}$ for all $t > a[t < a]$.

For odd n (e.g., for $n = 3$), we show that the oscillatory solutions of (1)–(1′) form an $(n-1)$-dimensional subspace, and that any nonoscillatory solution is asymptotic to the projectively unique positive solution as $t \to \infty$. Moreover, as t increases, only the positive and negative orthants $\pm\mathscr{P}$ are stable, and as t decreases *no* orthant is stable.

For even n, on the other hand, there exist *two* solutions of constant signature: the projectively unique positive solution $\mathbf{f}(t)$ and the projectively unique solution $\boldsymbol{\phi}(t)$ lying in $\mathscr{Q}$. The solutions which oscillate in both directions comprise an $(n-2)$-dimensional subspace; $\pm\mathscr{P}$ are the only stable orthants as t increases, and only $\pm\mathscr{Q}$ are stable as t decreases. Any nonoscillatory solution other than $\boldsymbol{\phi}$ and $\mathbf{f}$ is asymptotic to $\mathbf{f}$ as $t \to \infty$, and to $\boldsymbol{\phi}$ as $t \to -\infty$. More generally, $\boldsymbol{\phi}$ plays the same role for $s = -t$ as $\mathbf{f}$ does for t.[1]

LEMMA 1. The zeros of the components $x_i(t)$ of any nontrivial solution of the positive cyclic system (1) are isolated.

Proof ([2], p. 227). Suppose x_1, say, has a limit point t_0 of zeros. Then from (1), $\mathbf{x}(t_0) = \mathbf{0}$ by Rolle's theorem and continuity. Thus $\mathbf{x}(t) \equiv \mathbf{0}$.

LEMMA 2. If $\mathbf{x}(t)$ is any solution of (1), and $x_i(t)$ has r zeros on an interval (a, b), then $x_{i+k}(t)$ has at least $r-k$ zeros and at most $r+n-k$ zeros on (a, b).

Proof. This result follows from a repeated application of Rolle's theorem to (1), proceeding from x_i to x_{i+1}. We omit the details.

An immediate consequence is the

COROLLARY. All components of every oscillatory solution of a positive cyclic system have infinitely many zeros.

2. THE ALTERNATING ORTHANT

In [1], we demonstrated the projective uniqueness and other properties of a positive solution of (1)–(1′) for any n. In the case of even n a new phenom-

[1] Here and below, by a "u.p. system (1)," we mean a system which satisfies (1)–(1′). By "projectively unique," we mean unique up to a constant factor.

enon is made possible, namely the existence of a projectively unique solution $\boldsymbol{\phi}(t)$ in the alternating orthant $\mathcal{Q}$.

EXAMPLE 1. If (1) has constant coefficients, the linear transformations $t \mapsto (\Pi_1{}^n p_i)^{-1/n}\, t$ and $x_k \mapsto (\Pi_1^{k-1} p_i)^{-1}\, x_k$ reduce (1) to the normal form

$$dx_i/dt = x_{i+1}\,, \qquad i = 1, 2, \ldots, n. \tag{2}$$

For $n = 4$, the projectively unique positive solution, the projectively unique solution in $\mathcal{Q}$, and two oscillatory solutions are, respectively:

$$\mathbf{f}^* = e^t \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{\Phi}^* = e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{u}_1{}^* = \begin{pmatrix} \sin t \\ \cos t \\ -\sin t \\ -\cos t \end{pmatrix}, \quad \mathbf{u}_2{}^* = \begin{pmatrix} \cos t \\ -\sin t \\ -\cos t \\ \sin t \end{pmatrix}.$$

The corresponding solutions of the adjoint are, similarly,

$$\mathbf{g}^* = e^{-t} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{\psi}^* = e^{t} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_1{}^* = \begin{pmatrix} \sin t \\ \cos t \\ -\sin t \\ -\cos t \end{pmatrix}, \quad \mathbf{v}_2{}^* = \begin{pmatrix} \cos t \\ -\sin t \\ -\cos t \\ \sin t \end{pmatrix}.$$

Example 1 is representative in most important respects of the general even-dimensional u. p. system (1), and serves to illustrate many of the results to be proved below. To derive these, we require Theorems 1, 1′, and 4 of [1]. For convenience, we restate these known results as Theorems A, A′, and B, respectively.

THEOREM A. Any u. p. system (1) has a projectively unique positive solution $\mathbf{f}(t)$, i.e., one with all components positive everywhere in $(-\infty, +\infty)$.

A similar property is valid for the *adjoint* of (1)–(1′):

$$dy_i/dt = -p_{i-1}(t)\, y_{i-1}\,, \qquad i = 1, 2, \ldots, n. \tag{3}$$

THEOREM A′. The adjoint (3) of any u. p. system (1) has a projectively unique positive solution $\mathbf{g}(t)$.

The following lemma permits us to obtain corresponding results for solutions in $\mathcal{Q}$. Let the constant matrix $J = \operatorname{diag}\{1, -1, \ldots, 1, -1\}$, and let $P(t)$ represent the cyclic matrix of coefficients in (1). Thus, if

$$\mathbf{w} = (w_1\,, w_2\,, \ldots, w_n)^T,$$

then $J\mathbf{w} = (w_1\,, -w_2\,, \ldots, w_{n-1}\,, -w_n)^T$ when n is even.

LEMMA 3. If $d\mathbf{w}/dt = P(-t)\mathbf{w}(t)$ and n is even, then $J\mathbf{w}(-t)$ satisfies (1).

Proof. $(d/dt)\ J\mathbf{w}(-t) = -J\mathbf{w}'(-t) = -JP(t)\ \mathbf{w}(-t) = P(t)\ J\mathbf{w}(-t)$.
Observing that $P(-t)$ is u. p. and cyclic whenever $P(t)$ is, we obtain from Lemma 3 and Theorem A the following result.

THEOREM 1. If n is even, then the u. p. system (1) has a projectively unique solution $\boldsymbol{\phi}(t)$ in $\mathscr{Q}$; moreover, $\boldsymbol{\phi}(t) = J\mathbf{F}(-t)$, where $\mathbf{F}(t)$ is the projectively unique positive solution of $d\mathbf{w}/dt = P(-t)\,\mathbf{w}(t)$.

Similarly, the next result is an immediate consequence of Theorem A′ and Lemma 3.

THEOREM 1′. If n is even, the uniformly *negative* system (3) has a projectively unique solution $\boldsymbol{\psi}(t)$ in $\mathscr{Q}$.

Since the inner product $(\mathbf{x}(t), \mathbf{y}(t)) \equiv$ const. for every solution of (1) if, and only if, $\mathbf{y}(t)$ satisfies (3), the set of all solutions of (1) orthogonal to any fixed $\mathbf{y}(t)$ satisfying (3) forms a subspace $\mathbf{y}^{\perp}(t)$. Theorem 4 of [1] asserts

THEOREM B. Let $\mathbf{x}(t)$ be a solution and $\mathbf{f}(t)$ a positive solution of the u. p. system (1), and let $\mathbf{g}(t)$ be a positive solution of (3). Then the following conditions are equivalent: (i) $\mathbf{x}(t) = o[\mathbf{f}(t)]$ as $t \to \infty$; (ii) $\mathbf{x}(t) \in \mathbf{g}^{\perp}(t)$; (iii) $\mathbf{x}(t)$ is nowhere positive or negative; (iv) $\mathbf{x}(t)$ is not ultimately positive or negative.

THEOREM 2. For even n, let $\boldsymbol{\phi}(t)$ and $\boldsymbol{\psi}(t)$ be the projectively unique solutions in $\mathscr{Q}$ of the u. p. system (1) and (3), respectively. If $\mathbf{x}(t)$ is a solution of (1), then the following conditions are equivalent:

(i) $\mathbf{x}(t) = o(\boldsymbol{\phi})$ as $t \to -\infty$;
(ii) $\mathbf{x}(t) \in \boldsymbol{\psi}^{\perp}(t)$;
(iii) $\mathbf{x}(t) \in (\mathscr{Q} \cup -\mathscr{Q})'$ for all t;
(iv) $\mathbf{x}(t) \in (\mathscr{Q} \cup -\mathscr{Q})'$ ultimately as $t \to -\infty$.

If $\mathbf{y}(t)$ is a solution of (3), then the following conditions are equivalent:

(i) $\mathbf{y}(t) = o(\boldsymbol{\psi})$ as $t \to +\infty$;
(ii) $\mathbf{y}(t) \in \boldsymbol{\phi}^{\perp}(t)$;
(iii) $\mathbf{y}(t) \in (\mathscr{Q} \cup -\mathscr{Q})'$ for all t;
(iv) $\mathbf{y}(t) \in (\mathscr{Q} \cup -\mathscr{Q})'$ ultimately.

Proof. From the correspondence between the behavior of solutions in $\mathscr{Q}$ as t decreases and solutions in $\mathscr{P}$ as t increases, the first part is an immediate corollary of Theorem B. Now the adjoint system (3) can be made cyclic and u. p. if we replace t by $-t$ and permute the y_i's. This permutation preserves $\mathscr{Q} \cup -\mathscr{Q}$, and since the adjoint of (3) is (1), we obtain the second part of the theorem as a dual of the first part.

As in [1], this can be shown to imply the

COROLLARY. Suppose n is even. If $\mathbf{x}(t_0) \in \pm\mathscr{Q}$ for some t_0, where $\mathbf{x}(t)$ satisfies the u. p. system (1), then, for a suitable constant c, $\mathbf{x}(t) \sim c\boldsymbol{\phi}(t)$ as $t \to -\infty$. If $\mathbf{y}(t_1) \in \pm\mathscr{Q}$ for some t_1, where $\mathbf{y}(t)$ satisfies the uniformly *negative* system (3), then for a suitable constant c_1, $\mathbf{y}(t) \sim c_1\boldsymbol{\psi}(t)$ as $t \to +\infty$.

The following further description of the asymptotic behavior is immediately obtained from Lemma 3 as a corollary of the corresponding results in [1]. For even n, if $\mathbf{x}(t)$ is a solution of the u. p. system (1) which is never in $\mathscr{Q} \cup -\mathscr{Q}$, and $\mathbf{y}(t)$ is any solution such that $\mathbf{y}(t_1) \in \mathscr{Q}$, then $\mathbf{x}(t) = o[y(\mathbf{t})]$ as $t \to -\infty$.

Similarly, if $\mathbf{x}(t)$ is any solution of a u. p. cyclic system (1) with n even and $\mathbf{x}(t_0) \in \mathscr{Q}$ for some t_0, then for some constants c and C, and $i = 1, 2, \ldots, n$,

$$0 < ce^{-mt} \leqslant |x_i(t)| \leqslant Ce^{-Mt} \qquad \text{for all} \qquad t \leqslant t_0 .$$

This implies that the orthant $\mathscr{Q}$ is stable for decreasing t.

3. UNIFORM INSTABILITY

As in Section 1, a subset $\mathscr{S}$ of $\mathbf{x}$-space is called stable (as $t \uparrow$) under the action of (1) when, for any solution $\mathbf{x}(t)$ of the system, $\mathbf{x}(a) \in \mathscr{S}$ implies $\mathbf{x}(t) \in \mathscr{S}$ for all $t > a$. From the positiveness of the $p_i(t)$, it is clear that the positive orthant $\mathscr{P}$ and its negative $-\mathscr{P}$ are both stable under the action of (1). Also, $\mathscr{P}$ and $-\mathscr{P}$ are stable as $t \downarrow$ under the action of the adjoint (3) of any uniformly positive cyclic system (1).

For even n, positive cyclic systems (1) have another stable orthant (because the associated matrix is 2-cyclic). From obvious considerations, we find that for even n, the orthants $\mathscr{Q}$ and $-\mathscr{Q}$ are stable as $t \downarrow$ under the action of any positive cyclic system (1); they are also stable as $t \uparrow$ under the action of the *adjoint* of any positive cyclic system (1).

We now show that any orthant $\mathscr{O}$ other than $\pm\mathscr{P}$ and $\pm\mathscr{Q}$ is *unstable* for both increasing and decreasing t : no solution remains in $\mathscr{O}$ as $t \uparrow$ or $t \downarrow$.

LEMMA 4. Under the action of any u. p. system (1), all orthants other than $\pm\mathscr{P}$ and $\pm\mathscr{Q}$ are unstable as t increases and as t decreases.

Proof. Let $\mathscr{O}$ be any orthant other than $\pm\mathscr{P}$ and $\pm\mathscr{Q}$. If $n < 3$, the lemma is vacuously true. Then by cyclic symmetry and the linearity of (1), we may suppose without loss of generality that $\mathcal{O}$ has the signature $(-, +, +, \pm, \ldots)$. Let $\mathbf{x}(t)$ be any solution of (1) lying in $\mathcal{O}$ at time $t = a$, and assume that $\mathbf{x}(t) \in \mathcal{O}$ for all $t > a$. Then from (1), $0 > x_1(t) \uparrow$ const. for $t > a$, whence $\liminf_{t\to\infty} x_1'(t) = 0$. Thus, $\liminf_{t\to\infty} x_2(t) = 0$. But $0 < x_2 \uparrow$, a contradiction. Thus, $\mathcal{O}$ is unstable as $t \uparrow$. As $t \downarrow$ on the other hand, we may assume, again without loss in generality, that $\mathcal{O}$ has the signature $(-, -, +, \pm, \ldots)$. Replacing t by $-s$ in (1) and arguing as before, we conclude that $\mathcal{O}$ is unstable both for increasing and decreasing t.

From the projective uniqueness of the solutions in $\mathscr{P}$ and $\mathscr{Q}$, we obtain the

COROLLARY. Under the action of (1)–(1′), $\mathscr{P}$ and $-\mathscr{P}$ are the only orthants stable as $t \uparrow$, and $\mathscr{Q}$ and $-\mathscr{Q}$ are the only orthants stable as $t \downarrow$.

Therefore, the constant multiples of $\mathbf{f}(t)$ and $\boldsymbol{\phi}(t)$ are the only solutions of constant signature for all t.

We now strengthen Lemma 4 by proving that $\mathcal{O}$ is *uniformly unstable*, in the sense of

THEOREM 3. For any orthant $\mathcal{O}$ not $\pm\mathscr{P}$ or $\pm\mathscr{Q}$, there exists a $\tau = \tau(\mathcal{O})$ such that no solution of (1)–(1′) remains in $\bar{\mathcal{O}}$ for an interval of length τ.

Proof. We first observe that the residence time [of a solution of (1)–(1′)] on the boundary of any orthant is zero. More precisely, if

$$x_j(a) = x_{j+1}(a) = \cdots = x_k(a) = 0 \qquad \text{and} \qquad x_{k+1}(a) \neq 0,$$

then for $i = j, \ldots, k$, all $x_i(a^+)$ have the sign of $x_{k+1}(a)$ and all $x_i(a^-)$ have the opposite sign. Now from Lemma 4, for any particular initial $\mathbf{x}(0) = \mathbf{c} \in \bar{\mathcal{O}}$, there exists a $\tau = \tau(\mathbf{c}) > 0$ such that $\mathbf{x}(\tau)$ is in the interior of some orthant other than $\mathcal{O}$. By the continuous dependence of $\mathbf{x}(\tau)$ on $\mathbf{x}(0)$, it follows that this must be true for *all* $\mathbf{x}(0)$ in some (projective) neighborhood of $\mathbf{c}$. Finally, by the Heine-Borel theorem, it is possible to cover $\bar{\mathcal{O}}$ by a finite set of such neighborhoods. Then τ, the maximum of the associated residence times, has the required property.

4. Oscillation Theorems for n Odd

In this section, we shall generalize a number of the results of [1] for third-order uniformly positive cyclic systems (1)–(1′) to general odd-order u. p. systems (1). We first prove

THEOREM 4. For n odd, a nontrivial solution $\mathbf{x}(t)$ of the u. p. system (1) is oscillatory if, and only if, $\mathbf{x}(t) \in \mathbf{g}^{\perp}(t)$.

Proof. If $\mathbf{x}(t)$ is an oscillatory solution, then $\mathbf{x}(t) \in \mathbf{g}^{\perp}(t)$, from Theorem B. Conversely, suppose the nontrivial solution $\mathbf{x}(t)$ lies in $\mathbf{g}^{\perp}(t)$. Then $\mathbf{x}(t)$ can never be positive or negative and must therefore always lie in unstable orthants, from Lemma 4.

An immediate conclusion is

COROLLARY 1. For n odd, the oscillatory solutions form with the trivial solution $\mathbf{0}$ an $(n-1)$-dimensional subspace.

COROLLARY 2. For n odd, if $\mathbf{u}(t)$ is an oscillatory solution of the u. p. system (1) as $t\uparrow$, then $\mathbf{u}(t)$ is oscillatory as $t\downarrow$.

Proof. Since $\mathbf{u}(t) \in \mathbf{g}^{\perp}(t)$, from Lemma 4 $\mathbf{u}(t)$, for any t, lies in an orthant which is unstable as $t\downarrow$.

From Theorem 4, there are $n-1$ linearly independent oscillatory solutions in $\mathbf{g}^{\perp}$. This gives us

THEOREM 5. If n is odd, any $n-1$ linearly independent oscillatory solutions of the u. p. system (1)–(1′) constitute, together with $\mathbf{f}(t)$, a basis of solutions.

THEOREM 6. Let $\mathbf{f}(t)$ be a positive solution and let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be any nontrivial solutions of (1)–(1′) with n odd. Then $\mathbf{u}(t)$ oscillates if and only if $\mathbf{u} = o(\mathbf{f})$ as $t \to \infty$, and $\mathbf{v}(t)$ oscillates for $t\downarrow$ if, and only if, $\mathbf{f} = o(\mathbf{v})$ as $t \to -\infty$.

Proof. From Theorem 4, $\mathbf{u}$ oscillates if, and only if, $\mathbf{u} \in \mathbf{g}^{\perp}$. But from Theorem B, this is equivalent to $\mathbf{u} = o(\mathbf{f})$ as $t \to \infty$. Now suppose $\mathbf{v}$ oscillates as $t\downarrow$. Let $\mathbf{g}$ be a positive solution of the adjoint (3), and let $\mathbf{w}$ be a solution of (3) which oscillates as $t\downarrow$ and such that $(\mathbf{v}(0), \mathbf{w}(0)) \neq 0$. Then

$$(\mathbf{f}, \mathbf{g}) = |\,\mathbf{f}\,|\,|\,\mathbf{g}\,| \cos\alpha(t) \equiv \text{const.}$$
$$(\mathbf{v}, \mathbf{w}) = |\,\mathbf{v}\,|\,|\,\mathbf{w}\,| \cos\beta(t) \equiv \text{const.},$$

whence

$$|\,\mathbf{f}\,|/|\,\mathbf{v}\,| = \text{const.}\,|\,\mathbf{w}\,| \cos\beta/|\,\mathbf{g}\,| \cos\alpha.$$

But $\mathbf{w} = o(\mathbf{g})$ as $t \to -\infty$, from the dual to Theorem B. Furthermore, since $\mathbf{f}$ and $\mathbf{g}$ are bounded away from the coordinate planes (the proof in [1] for $n = 3$ being valid also for arbitrary n), $\cos\alpha \geqslant \epsilon > 0$ for all t. We conclude that $\mathbf{f} = o(\mathbf{v})$ as $t \to -\infty$.

Conversely, suppose $\mathbf{v}$ is a solution of (1)–(1′) such that $\mathbf{f} = o(\mathbf{v})$ as $t \to -\infty$. Then from Theorem 5 and Corollary 1 of Theorem 4, $\mathbf{v} = \mathbf{h} + c\mathbf{f}$ where $\mathbf{h}$ is an oscillatory solution as $t\uparrow$ and hence, from Corollary 2 of

Theorem 4, as $t \downarrow$. But $\mathbf{f} = o(\mathbf{h})$ as $t \to -\infty$, as shown above, whence $\mathbf{v}$ oscillates as $t \downarrow$.

An immediate consequence of Theorem 6 together with Theorem 5 is the

COROLLARY. Let $\mathbf{x}(t)$ be any solution of (1)–(1′) with n odd, and let $\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}$ be linearly independent oscillatory solutions. Then either $\mathbf{x}(t)$ is always positive or negative, or $\mathbf{x}(t) \sim \sum c_i \mathbf{u}_i(t)$ as $t \to -\infty$, for suitable c_i, whence $\mathbf{x}(t)$ oscillates for decreasing t.

Another immediate consequence is

THEOREM 7. Suppose n is odd. If $\mathbf{x}(t)$ is any nonoscillatory solution of (1)–(1′), then $\mathbf{x}(t) \sim c_1\mathbf{f}(t)$ as $t \to +\infty$, for a suitable constant c_1. If $\mathbf{x}(t)$ is a solution which does not oscillate as $t \to -\infty$, then $\mathbf{x}(t) = c_2\mathbf{f}(t)$ for a suitable c_2.

5. Oscillation Theorems for n Even

The facts are a little more complicated for even n because of the existence of the alternating orthant $\mathscr{Q}$. We shall show that not only is there a most rapidly growing solution $\mathbf{f}(t)$, but a most rapidly shrinking solution $\boldsymbol{\phi}(t)$ which is the projectively unique solution in $\mathscr{Q}$. Moreover, we shall obtain oscillation criteria, of which the first is

THEOREM 8. Consider the system (1)–(1′) for even n. A solution $\mathbf{x}(t)$ is oscillatory both as $t \uparrow$ and $t \downarrow$ if, and only if, $\mathbf{x}(t) \in \mathbf{g}^{\perp}(t) \cap \boldsymbol{\psi}^{\perp}(t)$. On the other hand, if a solution $\mathbf{x}(t)$ oscillates as $t \uparrow$ then $\mathbf{x}(t) \in \mathbf{g}(t)$; if it oscillates as $t \downarrow$, then $\mathbf{x}(t) \in \boldsymbol{\psi}^{\perp}(t)$.

Proof. The solution $\mathbf{x}(t)$ oscillates as $t \uparrow$ and $t \downarrow$ if, and only if, $\mathbf{x}(t)$ always lies in orthants which are unstable in both directions, i.e., if and only if $\mathbf{x}(t) \in (\mathscr{P} \cup -\mathscr{P})' \cap (\mathscr{Q} \cup -\mathscr{Q})'$. But this is equivalent to $\mathbf{x}(t) \in \mathbf{g}^{\perp}(t) \cap \boldsymbol{\psi}^{\perp}(t)$ from Theorems B and 2. A similar application of these theorems proves the remainder of the theorem.

Since there are $n - 2$ linearly independent oscillatory solutions in $\mathbf{g}^{\perp} \cap \boldsymbol{\psi}^{\perp}$ we immediately obtain the following two results.

COROLLARY. For n even, solutions of (1)–(1′) which are oscillatory in both directions form an $(n-2)$-dimensional subspace.

THEOREM 9. If n is even, any $n - 2$ linearly independent solutions of (1)–(1′) which oscillate as $t \uparrow$ and $t \downarrow$ constitute, together with $\mathbf{f}(t)$ and $\boldsymbol{\phi}(t)$, a basis of solutions.

THEOREM 10. Let $\mathbf{f}(t)$ be a positive solution of (1)–(1′) with n even, and $\boldsymbol{\phi}(t)$ a solution in $\mathscr{Q}$. If $\mathbf{u}(t)$ is an oscillatory solution as $t \uparrow$, then $\mathbf{u} = o(\mathbf{f})$ as $t \to +\infty$ and $\mathbf{f} = o(\mathbf{u})$ as $t \to -\infty$. If $\mathbf{v}(t)$ is a solution which oscillates as $t \downarrow$, then $\boldsymbol{\phi} = o(\mathbf{v})$ as $t \to +\infty$ and $\mathbf{v} = o(\boldsymbol{\phi})$ as $t \to -\infty$.

The proof is similar to that of Theorem 6.

COROLLARY. For n even, the projectively unique solution $\boldsymbol{\phi}(t)$ in $\mathscr{Q}$ is the most rapidly shrinking solution of (1)–(1′); i.e., if $\mathbf{x}(t)$ is any solution other than $k\boldsymbol{\phi}$ for any k, then $\boldsymbol{\phi} = o(\mathbf{x})$ as $t \uparrow$. Similarly, the projectively unique positive solution $\mathbf{f}(t)$ is the most rapidly growing solution as $t \uparrow$, and the most rapidly shrinking one as $t \downarrow$.

Proof. From Theorem 9, for suitable constant coefficients

$$\mathbf{x} = a_1\mathbf{u}_1 + \cdots + a_{n-2}\mathbf{u}_{n-2} + a_{n-1}\mathbf{f} + k\boldsymbol{\phi}$$

for linearly independent solutions $\mathbf{u}_1, \ldots, \mathbf{u}_{n-2}$ which oscillate in both directions. But as $t \uparrow$, $\boldsymbol{\phi} = o(\mathbf{u}_i)$ and $\mathbf{u}_i = o(\mathbf{f})$, $i = 1, 2, \ldots, n-2$. Therefore, $\boldsymbol{\phi} = o(\mathbf{x})$ unless $a_1 = \cdots = a_{n-1} = 0$. The rest of the proof is similar.

As a consequence of the preceding two theorems, we obtain

THEOREM 11. Consider the system (1)–(1′) with n even. If $\mathbf{x}(t)$ is any nonoscillatory solution, then either $\mathbf{x}(t) = c_0\boldsymbol{\phi}(t)$ or $\mathbf{x}(t) \sim c_1\mathbf{f}(t)$ as $t \uparrow$, for a suitable constant c_0 or c_1. If $\mathbf{x}(t)$ is any solution which does not oscillate as $t \downarrow$, then either $\mathbf{x}(t) = c_3\mathbf{f}(t)$ or $\mathbf{x}(t) \sim c_4\boldsymbol{\phi}(t)$ as $t \downarrow$, for suitable c_3 or c_4.

Proof. Suppose $\mathbf{x}(t) \neq c_0\boldsymbol{\phi}(t)$ is any nonoscillatory solution of (1)–(1′) with n even. Then

$$\mathbf{x} = a_1\mathbf{u}_1 + \cdots + a_{n-2}\mathbf{u}_{n-2} + c_0\boldsymbol{\phi} + c_1\mathbf{f},$$

where the $\mathbf{u}_i$'s are linearly independent solutions which are oscillatory in both directions, and the coefficient c_1 cannot be zero. For if $c_1 = 0$, then from the Corollary to Theorem 10, $\boldsymbol{\phi} = o(a_i\mathbf{u}_i)$ for at least one i with $a_i \neq 0$, whence $\mathbf{x} \sim \sum a_i\mathbf{u}_i$ as $t \uparrow$ and would then oscillate. Consequently, $\mathbf{x} \sim c_1\mathbf{f}$ since $\boldsymbol{\phi} = o(\mathbf{f})$ and $\mathbf{u}_i = o(\mathbf{f})$ as $t \uparrow$. The rest of the proof is similar.

6. FOURTH ORDER SYSTEMS

In [1] we saw that if $\mathbf{x}(t)$ is an oscillatory solution of (1)–(1′) for $n = 3$, then the signs of its components permute cyclically as t increases, each sign change affecting the second of the two components having like sign. Thus, the signature of $\mathbf{x}(t)$ changes cyclically in six phases. We shall show that for $n = 4$, the signature of an oscillatory solution $\mathbf{x}(t)$ of (1)–(1′) changes cyclically in *eight* phases.

Since the positive orthant $\mathscr{P}$ is stable for $t \uparrow$, the signs of the components of any oscillatory solution $\mathbf{x}(t)$ of (1)–(1′) for $n = 4$ cannot all be the same. Since $\mathscr{Q}$ is stable for $t \downarrow$, once $\mathbf{x}(t)$ leaves $\mathscr{Q}$ it cannot reenter, so ultimately the signs of its components cannot alternate. Then by cyclic symmetry and multiplication by -1, we may assume without loss in generality that at an arbitrary but fixed large t_0, the signature of $\mathbf{x}(t_0)$ is either $(+, +, -, +)$ or $(+, +, -, -)$. In the first case, from elementary considerations as in ([1], Theorem 7), $x_2(t)$ is the first component to vanish for $t > t_0$. In the second case, either x_2 or x_4 may vanish first, resulting in a signature which is identical to the first case to within cyclic symmetry and multiplication by -1. This results in

THEOREM 12. If $\mathbf{x}(t)$ is any oscillatory solution of (1)–(1′) with $n = 4$, as $t \uparrow$ each change in sign occurs at the last component of a sequence of components of like sign. The resulting flow diagram of signatures then occurs as follows:

$$(+, +, -, -) \to \begin{Bmatrix} (+, -, -, -) \\ (+, +, -, +) \end{Bmatrix} \to (+, -, -, +) \to \begin{Bmatrix} (-, -, -, +) \\ (+, -, +, +) \end{Bmatrix} \to$$

$$(-, -, +, +) \to \begin{Bmatrix} (-, +, +, +) \\ (-, -, +, -) \end{Bmatrix} \to (-, +, +, -) \to \begin{Bmatrix} (+, +, +, -) \\ (-, +, -, -) \end{Bmatrix} \to$$

etc., where either signature in any bracketed term may occur.

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